NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 1

2023 Year 12 Course Assessment Task 4 (Trial Examination)
Wednesday August 9th, 2023

General instructions

- Working time – 2 hours. (plus 10 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided (on page 11)

SECTION II

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

NESA STUDENT #: .... **# BOOKLETS USED:** .....

Class (please ✔)

◯ 12MAX.1 – Miss C. Kim
◯ 12MAX.2 – Miss Lee
◯ 12MAX.3 – Miss J. Kim
◯ 12MXX.1 – Mr Ho
◯ 12MXX.2 – Mr Sekaran
◯ 12MXX.3 – Ms Ham

Marker's use only.

QUESTION	1-10	11	12	13	14	**Total**	%
MARKS	$\overline{10}$	$\overline{15}$	$\overline{17}$	$\overline{17}$	$\overline{11}$	$\overline{70}$	

Section I

10 marks
Attempt Question 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer grid provided (labelled as page 11).

Questions **Marks**

1. What is the range of $y = 2\cos^{-1} x$? **1**

(A) $-2 \le y \le 2$ (B) $-\pi \le y \le \pi$ (C) $0 \le y \le \pi$ (D) $0 \le y \le 2\pi$

2. It is given that $f(x) = e^{x+2}$. Which of the following is the inverse function $f^{-1}(x)$? **1**

(A) $f^{-1}(x) = -2 + \log_e x$

(B) $f^{-1}(x) = 2 + \log_e x$

(C) $f^{-1}(x) = e^{x-2}$

(D) $f^{-1}(x) = e^{x+2}$

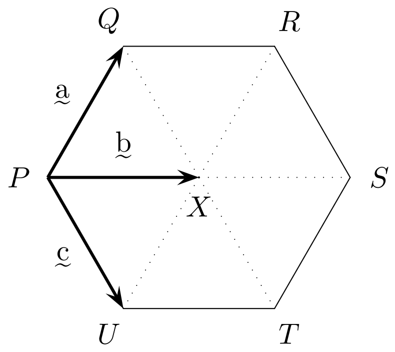
3. How many numbers greater than 2100 can be formed with the digits $1, 2, 3, 4$ if no digit is to be used more than once? **1**

(A) 16 (B) 18 (C) 24 (D) 48

4. How many solutions does the equation $x^{\frac{1}{3}} = |x-2| - 3$ have? **1**

(A) 0 (B) 1 (C) 2 (D) 3

5. $PQRSTU$ is a regular hexagon with side lengths of 4 cm and is divided into six equilateral triangles. It is given that $\overrightarrow{PQ} = \underset{\sim}{a}$, $\overrightarrow{PX} = \underset{\sim}{b}$ and $\overrightarrow{PU} = \underset{\sim}{c}$, as shown in the diagram below. **1**



Which of the following is the value of $\underset{\sim}{a} \cdot (\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c})$?

(A) 8 (B) 16 (C) 32 (D) 48

6. Which of the following expressions is equal to $\sin x - 3\cos x$? **1**

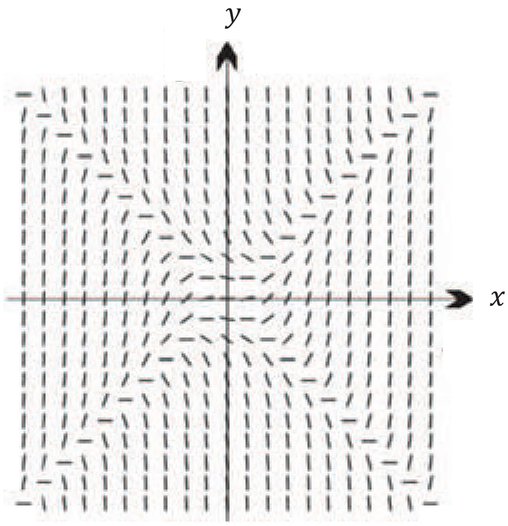
(A) $\sqrt{10}\,\cos(x + \tan^{-1} 3)$

(B) $\sqrt{10}\,\sin(x - \tan^{-1} 3)$

(C) $\sqrt{10}\,\cos\left(x + \tan^{-1}\dfrac{1}{3}\right)$

(D) $\sqrt{10}\,\sin\left(x - \tan^{-1}\dfrac{1}{3}\right)$

7. Which of the following differential equations best corresponds to the slope field shown below? **1**



(A) $y' = \dfrac{1}{4}(-x^2 + y^2)$

(B) $y' = -\dfrac{1}{4}(x^2 + y^2)$

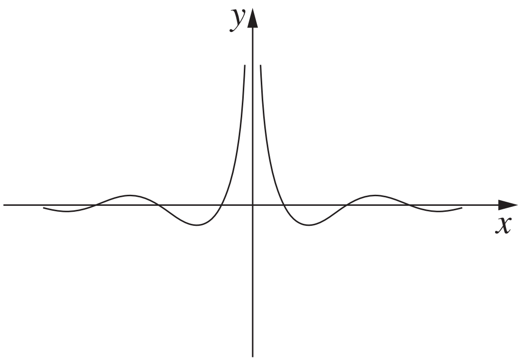
(C) $y' = \dfrac{1}{4}(x^2 - y^2)$

(D) $y' = \dfrac{1}{4}(x^2 + y^2)$

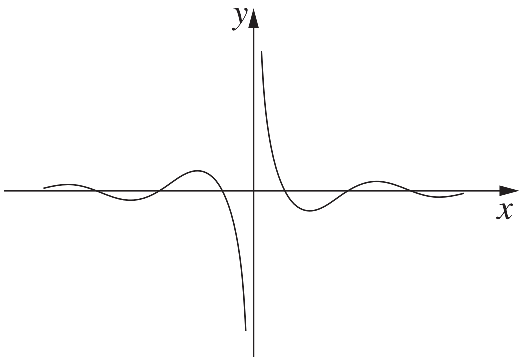
Examination continues overleaf...

8. Which diagram best represents the graph $y = \dfrac{\cos x}{x}$? **1**

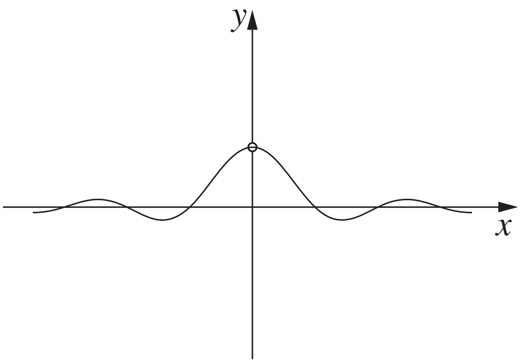
(A)



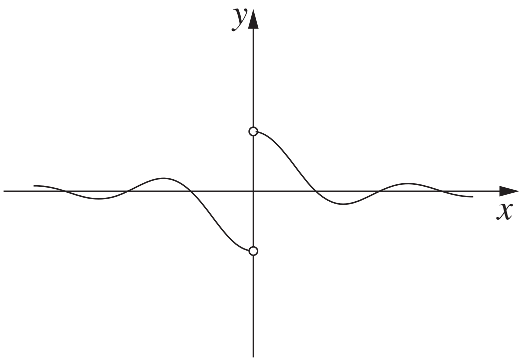
(C)



(B)



(D)



9. A particle is moving in a straight line such that its displacement, x, from the origin after t seconds is given by $x = -3\cos^2 t$. **1**

Which of the following best describes the motion of the particle when $t = \dfrac{5\pi}{6}$

(A) The particle is moving in the positive direction with an increasing speed

(B) The particle is moving in the positive direction with a decreasing speed

(C) The particle is moving in the negative direction with an increasing speed

(D) The particle is moving in the negative direction with a decreasing speed

10. Consider the expansion of **1**

$$\left(1 + x + x^2 + \cdots + x^n\right)\left(1 + 2x + 3x^2 + \cdots + (n+1)x^n\right)$$

What is the coefficient of x^n when $n = 50$?

(A) 1225 (B) 1300 (C) 1326 (D) 1431

Section II

60 marks
Attempt Questions 11 to 14
Allow approximately 1 hours and 45 minutes for this section.

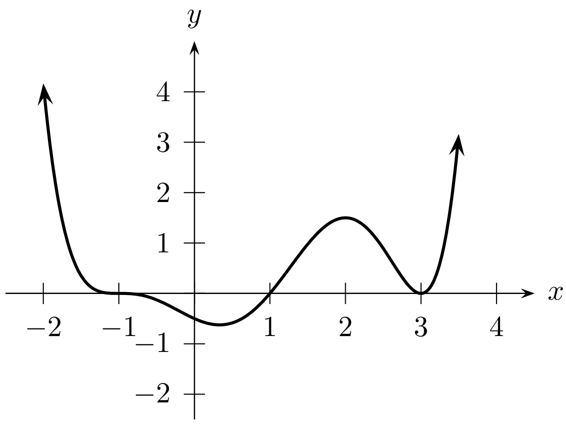
Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Commence a NEW booklet. **Marks**

(a) If $f(x) = \tan^{-1}\left(\frac{x}{2}\right)$, find the value of $f'(2)$. **2**

(b) It is given that $\cos 2A = \frac{2}{5}$. Find the value of $\sin A \sin 5A + \cos A \cos 5A$. **2**

(c) The graph of the polynomial function $y = P(x)$ is given below.



i. Without using calculus, write down a possible equation of $y = P(x)$. **1**

ii. Solve $P(x) \leq 0$ for x. **2**

(d) i. By considering $(1 + x)^{2n}$, or otherwise, show that **2**

$$\sum_{k=0}^{2n} \binom{2n}{k} \left(-\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^{2n}$$

ii. Hence evaluate **2**

$$\sum_{k=0}^{2n-1} \binom{2n}{k} \left(-\frac{1}{2}\right)^k$$

(e) There are twelve people that need to be divided into groups.

i. In how many ways can they be divided into two groups of six people? **1**

ii. In how many ways can they be divided into two groups of six people if two particular people must be in the same group? **1**

iii. In how many ways can they be seated around one circular table, if the two groups of six people from (i) must be seated among their own groups? **2**

Examination continues overleaf...

Question 12 (17 Marks) Commence a NEW booklet. **Marks**

(a) Consider the polynomial $P(x) = ax^7 + bx^6 + 1$, where $a, b \in \mathbb{R}$ and $a, b \neq 0$. Find the values of a and b when $P(x)$ is divisible by $(x-1)^2$. **3**

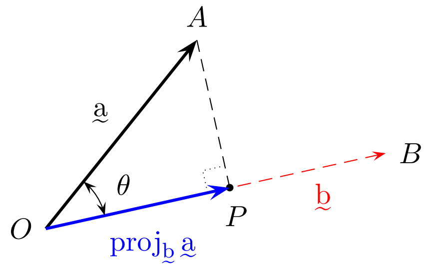
(b) It is given that $t = \tan A$ and $p \cos 2A - \sin 2A = 1$, where $p \in \mathbb{R}$. Show that **3**

$$t = \frac{p-1}{p+1}$$

where $0 \leq A \leq \frac{\pi}{2}$.

(c) By using the substitution $x = \frac{1}{4}(u^2+5)$, find $\int 3x\sqrt{4x-5}\,dx$. **3**

(d) The diagram below shows $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$ and θ is the acute angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$.



i. By using the fact that $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos\theta$, or otherwise, prove that **2**

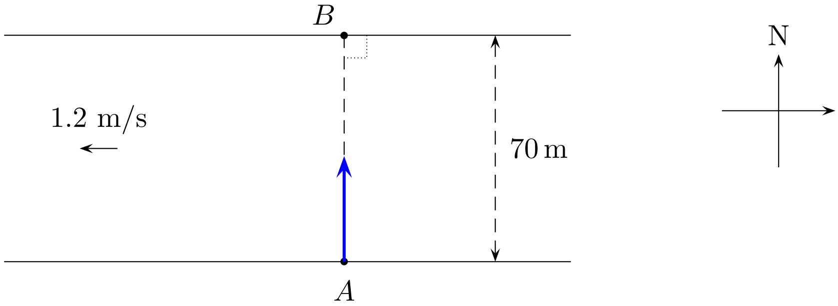
$$\text{proj}_{\underset{\sim}{b}}\,\underset{\sim}{a} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\,\underset{\sim}{b}$$

It is given that $\underset{\sim}{a} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$.

ii. Find a perpendicular vector to $\underset{\sim}{b}$. **1**

iii. Hence, or otherwise, find the shortest distance from the point A to the line OB by using vector methods. **2**

(e) A fish at point A wants to swim across the river that is 70 m wide. The banks of the river are parallel and the points A and B are on opposite sides of the river. **3**



The fish swims at a speed of 1.8 m/s and the river is flowing downstream at 1.2 m/s. However, a 0.5 m/s wind blows constantly to the north-westerly direction.

What is the bearing that the fish should head to in order to land directly at point B from point A?

Examination continues overleaf...

Question 13 (17 Marks) Commence a NEW booklet. **Marks**

(a) Prove by mathematical induction that for all positive integer values of n, **3**

$$\left(1^2 \times 2^1\right) + \left(2^2 \times 2^2\right) + \left(3^2 \times 2^3\right) + \cdots + \left(n^2 \times 2^n\right) = \left(n^2 - 2n + 3\right) \times 2^{n+1} - 6$$

(b) The temperature T_1 of a beaker of chemical, and the temperature T_2 of a surrounding barrel of cooler water, satisfy the following equations:

$$\frac{dT_1}{dt} = -k\left(T_1 - T_2\right)$$

$$\frac{dT_2}{dt} = \frac{3}{4}k\left(T_1 - T_2\right)$$

where k is a positive constant.

i. Show, by differentiation, that $\frac{3}{4}T_1 + T_2 = C$, where C is a constant. **1**

ii. Hence, show that $\frac{dT_1}{dt} = kC - \frac{7}{4}kT_1$. **1**

iii. By integration, show that $T_1 = \frac{4}{7}C + Be^{-\frac{7}{4}kt}$, where B is a constant, is a solution to the differential equation $\frac{dT_1}{dt}$ in (ii). **3**

Initially, the beaker of chemical had a temperature of $T_1 = 120°C$ and the barrel of water had a temperature of $T_2 = 22°C$. Ten minutes later, the temperature of the beaker of chemical had fallen to $90°C$.

iv. Show that **2**

$$T_1 = 64 + 56e^{-\frac{7}{4}kt}$$

v. Show that eventually the beaker of chemical and the surrounding barrel of water reach the same temperature. **2**

Mr Kim found out that the number of molecules N in the beaker of chemical is inversely proportional to the temperature of the beaker, T_1, which is given by $N = \frac{1000}{T_1}$.

vi. Show that the rate of growth of N is **3**

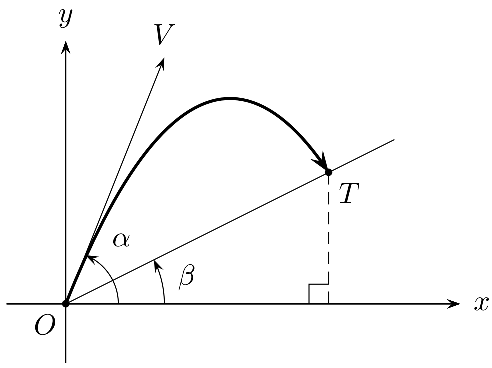
$$\frac{dN}{dt} = \frac{7k}{4}N\left(1 - \frac{64}{1000}N\right)$$

vii. Hence, or otherwise, find the maximum rate of growth of N given that $k = 0.001$. **2**

Question 14 (11 Marks) Commence a NEW booklet. **Marks**

(a) The diagram shows an inclined plane that makes an angle of β with the horizontal. A projectile is fired from O, at the bottom of the incline, with a speed of V m/s at an angle of elevation α to the horizontal, as shown below. **3**

The projectile hits the inclined plane at T.



The displacement vector, t seconds after projection, is given by

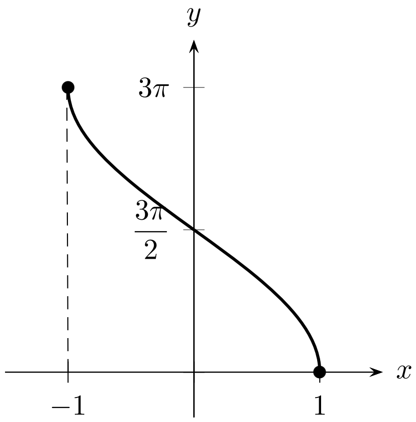
$$\underset{\sim}{r} = (Vt\cos\alpha)\,\underset{\sim}{i} + (-5t^2 + Vt\sin\alpha)\,\underset{\sim}{j}$$

(Do NOT prove this.)

Show that the time taken to hit the inclined plane is

$$t = \frac{V}{5}(\sin\alpha - \cos\alpha\tan\beta)$$

(b) Below is the graph of the function $y = 3\cos^{-1} x$. **4**



Find the exact volume of the solid of revolution formed if the area bounded by $y = 3\cos^{-1} x$, the coordinate axes and the line $x = -1$ is rotated about the y-axis.

Examination continues overleaf...

(c) Consider the following functions: **4**

$$y = \sin x \cos x$$

$$y = mx$$

where $m < 0$.

It is given that there is only one point of intersection at $x = x_0$ between the two functions for $x > 0$.

Show that there exists another value $x = x_1$, such that $x_1 \in \left(\pi, \dfrac{3\pi}{2}\right)$ and

$$m = \cos x_1 \quad \text{and} \quad \tan x_1 = x_1$$

(*Hint*: Draw picture)

End of paper.

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

NESA STUDENT #: ....

Class (please ✔)

◯ 12MXX.1 – Ms C.Kim

◯ 12MXX.2 – Ms Lee

◯ 12MAX.3 – Mr Lam/ Ms J.Kim

◯ 12MXX.1 – Mr Ho

◯ 12MXX.2 – Mr Sekaran

◯ 12MXX.3 – Ms Ham

Directions for multiple choice answers

- Read each question and its suggested answers.
- Select the alternative Ⓐ, Ⓑ, Ⓒ, or Ⓓ that best answers the question.
- Mark only one circle per question. There is only *one* correct choice per question.
- Fill in the response circle completely, using blue or black pen, e.g.

  Ⓐ Ⓑ ● Ⓓ

- If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

  Ⓐ Ⓑ ~~●~~ ●

- If you continue to change your mind, write the word `correct` and clearly indicate your final choice with an arrow as shown below:

  Ⓐ Ⓑ ~~●~~ ~~●~~ correct

1 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ
2 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ
3 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ
4 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ
5 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ
6 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ
7 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ
8 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ
9 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ
10 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ

Sample Band 6 Responses

Section I

1. (D) **2.** (A) **3.** (B) **4.** (C) **5.** (B)
6. (B) **7.** (C) **8.** (C) **9.** (D) **10.** (C)

Section II

Question 11

(a) (2 marks)

✓ [1] for $f'(x)$
✓ [1] for final answer

$$\begin{aligned} f'(x) &= \frac{2}{x^2+4} \\ f'(2) &= \frac{2}{8} = \frac{1}{4} \end{aligned}$$

(b) (2 marks)

✓ [1] for $\cos 4A$
✓ [1] for final answer

$$\begin{aligned} \sin A \sin 5A + \cos A \cos 5A &= \cos 5A \cos A + \sin 5A + \sin A \\ &= \cos(5A - A) \\ &= \cos 4A \\ &= 2\cos^2 2A - 1 \\ &= 2\left(\frac{2}{5}\right)^2 - 1 \\ &= -\frac{17}{25} \end{aligned}$$

(c) i. (1 mark)

$$P(x) = (x+1)^3(x-1)(x-3)^2$$

ii. (2 marks)

✓ [1] for $-1 \leq x \leq 1$
✓ [1] for final answer

$$-1 \leq x \leq 1 \text{ or } x = 3$$

(d) i. (2 marks)

✓ [1] for expanding $(1+x)^{2n}$
✓ [1] for final result

$$(1+x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^k$$

Let $x = -\dfrac{1}{2}$

$$\begin{aligned}\sum_{k=0}^{2n} \binom{2n}{k} \left(-\frac{1}{2}\right)^k &= \left(1-\frac{1}{2}\right)^{2n}\\ &= \left(\frac{1}{2}\right)^{2n}\end{aligned}$$

ii. (1 mark)

$$\begin{aligned}\sum_{k=0}^{2n-1} \binom{2n}{k} \left(-\frac{1}{2}\right)^k &= \sum_{k=0}^{2n} \binom{2n}{k} \left(-\frac{1}{2}\right)^k - \binom{2n}{2n} \left(-\frac{1}{2}\right)^{2n}\\ &= \left(\frac{1}{2}\right)^{2n} - \left(\frac{1}{4}\right)^n\\ &= 0\end{aligned}$$

(e) i. (1 mark)

$$\frac{{}^{12}C_6 \times {}^6 C_6}{2} = 462 \text{ ways}$$

ii. (1 mark)

$${}^{10}C_4 \times {}^6 C_6 = 210 \text{ ways}$$

iii. (2 marks)

$$6! \times 6! = 518400 \text{ ways}$$

Question 12

(a) (3 marks)

✓ [1] for $P(1) = P'(1) = 0$

✓ [1] for (1) and (2)

✓ [1] for final answer

$$\begin{aligned}P(x) &= ax^7 + bx^6 + 1\\ P'(x) &= 7ax^6 + 6bx^5\end{aligned}$$

$\because x = 1$ is a double root, $P(1) = P'(1) = 0$.

$$\begin{aligned}a + b + 1 &= 0 \quad \ldots(1)\\ 7a + 6b &= 0 \quad \ldots(2)\end{aligned}$$

(2) − (1) × 6,

$$a = 6 \text{ and } b = -7$$

(b) (3 marks)

- ✓ [1] for substituting into $\cos 2A$ and $\sin 2A$
- ✓ [1] for making substantial progress
- ✓ [1] for final result

If $t = \tan A$, $\cos 2A = \dfrac{1-t^2}{1+t^2}$ and $\sin 2A = \dfrac{2t}{1+t^2}$.

$$\begin{aligned} p\cos 2A - \sin 2A &= 1 \\ p \times \frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} &= 1 \\ p(1-t^2) - 2t &= 1+t^2 \\ p(1-t^2) &= t^2+2t+1 \\ p(1-t)(1+t) &= (t+1)^2 \\ p(1-t) &= 1+t \quad \ldots \left(\because \tan A > 0 \text{ for } 0 \le A \le \frac{\pi}{2},\ 1+t \neq 0\right) \\ t(1+p) &= p-1 \\ \therefore t &= \frac{p-1}{p+1} \end{aligned}$$

(c) (3 marks)

- ✓ [1] for $dx = \frac{u}{2}du$
- ✓ [1] for substitution and integration
- ✓ [1] for final answer

$$\begin{aligned} x &= \frac{1}{4}(u^2+5), \quad u^2 = 4x-5, \quad u = \sqrt{4x-5} \\ dx &= \frac{1}{4} \times 2u\,du = \frac{u}{2}\,du \end{aligned}$$

$$\begin{aligned} \therefore \int 3x\sqrt{4x-5}\,dx &= \frac{3}{4}\int (u^2+5)\sqrt{u^2} \times \frac{u}{2}\,du \\ &= \frac{3}{8}\int u^4 + 5u^2\,du \\ &= \frac{3}{8}\left(\frac{u^5}{5} + 5 \times \frac{u^3}{3}\right) + c \\ &= \frac{3}{40}\left(\sqrt{4x-5}\,\right)^5 + \frac{5}{8}\left(\sqrt{4x-5}\,\right)^3 + c \\ &= \frac{3}{40}(4x-5)^{\frac{5}{2}} + \frac{5}{8}(4x-5)^{\frac{3}{2}} + c \end{aligned}$$

(d) i. (2 marks)

- ✓ [1] for (*)
- ✓ [1] for final result

In $\triangle AOP$, $\quad \cos\theta = \dfrac{OP}{|\underset{\sim}{a}|}$

$$\begin{aligned} \text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{a} &= OP\ \hat{\underset{\sim}{b}} \quad \ldots (*) \\ &= |\underset{\sim}{a}| \cos\theta\ \hat{\underset{\sim}{b}} \end{aligned}$$

$\because \cos\theta = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|}$,

$$\begin{aligned} \text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{a} &= \cancel{|\underset{\sim}{a}|}\ \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\cancel{|\underset{\sim}{a}|}\,|\underset{\sim}{b}|}\ \hat{\underset{\sim}{b}} \\ &= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|}\ \frac{\underset{\sim}{b}}{|\underset{\sim}{b}|} \\ &= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|^2}\ \underset{\sim}{b} \\ &= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\ \underset{\sim}{b} \end{aligned}$$

ii. (1 mark)

$$\begin{pmatrix} -6 \\ 3 \end{pmatrix} \text{ or } \begin{pmatrix} -2 \\ 1 \end{pmatrix} \text{ or any other vector that is perpendicular}$$

iii. (2 marks)

✓ [1] for $\text{proj}_{\underset{\sim}{a}}\, \underset{\sim}{u}$

✓ [1] for final answer

Let $\underset{\sim}{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

$$\begin{aligned} \text{proj}_{\underset{\sim}{a}}\, \underset{\sim}{u} &= \frac{\begin{pmatrix} -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix}}{\sqrt{(-2)^2 + 1^2}}\ \hat{\underset{\sim}{u}} \\ &= \frac{4+5}{\sqrt{5}} \hat{\underset{\sim}{u}} \\ &= \frac{9}{\sqrt{5}} \hat{\underset{\sim}{u}} \end{aligned}$$

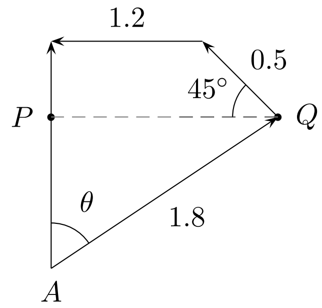
$\therefore$ The shortest distance from A to the line OB is $\left|\text{proj}_{\underset{\sim}{a}}\, \underset{\sim}{u}\right| = \dfrac{9}{\sqrt{5}}$.

(e) (3 marks)

✓ [1] for vector diagram or correct vertical vector

✓ [1] for PQ, or equivalent merit

✓ [1] for final answer



$$PQ = 1.2 + \frac{0.5}{\sqrt{2}} = \frac{6}{5} + \frac{1}{2\sqrt{2}}$$

In $\triangle APQ$,

$$\sin\theta = \frac{\frac{6}{5} + \frac{1}{2\sqrt{2}}}{1.8}$$
$$\theta = 59.6647...^\circ$$

$\therefore$ The bearing that the fish should head to is 060° T (to the nearest degree)

Question 13

(a) (3 marks)

- ✓ [1] for proving the base case
- ✓ [1] for use of the assumption
- ✓ [1] for final result

Let $P(n)$ be the given proposition.

- Prove $P(1)$ is true

$$\begin{aligned}\text{LHS} &= 1^2 \times 2^1 = 2\\ \text{RHS} &= (1^2 - 2 + 3) \times 2^2 - 6 = 2\end{aligned}$$

$\therefore$ LHS= RHS, $P(n)$ is true for $n = 1$.

- Assume $P(k)$ is true

$$\left(1^2 \times 2^1\right) + \left(2^2 \times 2^2\right) + \left(3^2 \times 2^3\right) + \cdots + \left(k^2 \times 2^k\right) = (k^2 - 2k + 3) \times 2^{k+1} - 6$$

- Prove $P(k+1)$ is true

$$\begin{aligned}(1^2 \times 2^1) + (2^2 \times 2^2) + \cdots + \left((k+1)^2 \times 2^{k+1}\right) &= \left((k+1)^2 - 2(k+1) + 3\right) \times 2^{k+2} - 6\\ &= (k^2 + 2k + 1 - 2k - 2 + 3) \times 2^{k+2} - 6\\ &= \left(k^2 + 2\right) \times 2^{k+2} - 6\end{aligned}$$

$$\begin{aligned}
\text{LHS} &= (k^2-2k+3)\times 2^{k+1} - 6 + (k+1)^2 \times 2^{k+1} \quad \ldots \text{ from the assumption}\\
&= 2^{k+1}(k^2-2k+3+k^2+2k+1)-6\\
&= 2^{k+1}\times 2(k^2+2)-6\\
&= (k^2+2)\times 2^{k+2}-6\\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k+1)$ is true.

- Conclusion
  By mathematical induction, the proposition is true for all positive integers.

(b) i. (1 mark)

$$\begin{aligned}
\frac{d}{dt}\left(\frac{3}{4}T_1+T_2\right) &= \frac{3}{4}\frac{dT_1}{dt}+\frac{dT_2}{dt}\\
&= -\frac{3}{4}k(T_1-T_2)+\frac{3}{4}k(T_1-T_2)\\
&= 0
\end{aligned}$$

$\therefore \dfrac{3}{4}T_1+T_2 = C$, where C is a constant.

ii. (1 mark)

$$\because T_2 = C-\frac{3}{4}T_1,$$

$$\frac{dT_1}{dt} = -k\left(T_1-\left(C-\frac{3}{4}T_1\right)\right) = -k\left(T_1-C+\frac{3}{4}T_1\right) = -k\left(\frac{7}{4}T_1-C\right) = kC-\frac{7}{4}kT_1$$

iii. (3 marks)

✓ [1] for separation of variables and integration

✓ [1] substantial progress in making T_1 the subject

✓ [1] for final result

$$\begin{aligned}
\frac{dT_1}{dt} &= kC-\frac{7}{4}kT_1\\
\int \frac{1}{C-\frac{7}{4}T_1}\,dT_1 &= \int k\,dt\\
-\frac{4}{7}\ln\left|C-\frac{7}{4}T_1\right| &= kt+A \quad \ldots \text{ where } A \text{ is a constant}\\
\ln\left|C-\frac{7}{4}T_1\right| &= -\frac{7}{4}kt+A_1\\
C-\frac{7}{4}T_1 &= \pm\, e^{-\frac{7}{4}kt+A_1}\\
&= A_2\,e^{-\frac{7}{4}kt}\\
\therefore \frac{7}{4}T_1 &= C-A_2\,e^{-\frac{7}{4}kt}\\
T_1 &= \frac{4}{7}C-A_3\,e^{-\frac{7}{4}kt} \quad \ldots \text{ where } A_1, A_2, A_3 \text{ are constants}\\
\therefore T_1 &= \frac{4}{7}C+B\,e^{-\frac{7}{4}kt} \quad \ldots \text{ where } B \text{ is a constant}
\end{aligned}$$

iv. (2 marks)

✓ [1] for substitution of $t = 0$, $T_1 = 120$ *or* the value of C

✓ [1] for B

When $t = 0$, $T_1 = 120°C$ and $T_2 = 22°C$

$$C = \frac{3}{4}T_1 + T_2 = \frac{3}{4} \times 120 + 22 = 112$$

and

$$\begin{aligned} T_1 &= \frac{4}{7} \times 112 + B\,e^{-\frac{7}{4}kt} \\ 120 &= 64 + B \\ \therefore B &= 56 \end{aligned}$$

$$\therefore T_1 = 64 + 56e^{-\frac{7}{4}kt}$$

v. (2 marks)

✓ [1] for showing $T_1 \to 64$

✓ [1] for showing $T_2 \to 64$

As $t \to \infty$, $e^{-\frac{7}{4}kt} \to 0$

$$\therefore T_1 \to 64$$

$$\because T_2 = C - \frac{3}{4}\,T_1,$$

As $T_1 \to 64$,

$$\begin{aligned} T_2 &\to 112 - \frac{3}{4} \times 64 \\ &= 64 \end{aligned}$$

$\therefore$ The beaker and the surrounding barrel eventually reach the same temperature.

vi. (3 marks)

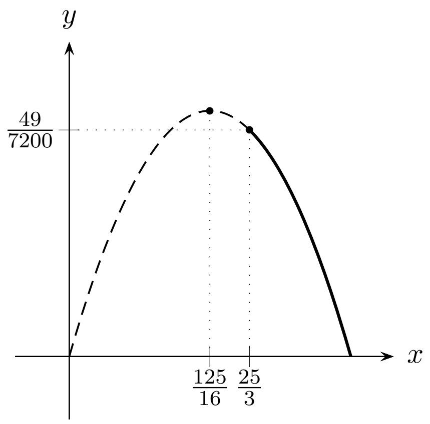
✓ [1] for $\dfrac{dN}{dt}$

✓ [1] for substantial progress in manipulation of $\dfrac{dN}{dt}$

✓ [1] for final result

$$\begin{aligned}
N &= \frac{1000}{64+56e^{-\frac{7}{4}kt}} \\
\frac{dN}{dt} &= -\frac{1000}{\left(64+56e^{-\frac{7}{4}kt}\right)^2} \times \left(-\frac{7}{4}k \times 56\, e^{-\frac{7}{4}kt}\right) \\
&= \frac{1000}{64+56e^{-\frac{7}{4}kt}} \times \frac{7}{4}k \times \frac{56k\, e^{-\frac{7}{4}kt}}{64+56e^{-\frac{7}{4}kt}} \\
&= N \times \frac{7k}{4} \times \frac{64+56e^{-\frac{7}{4}kt}-64}{64+56e^{-\frac{7}{4}kt}} \\
&= N \times \frac{7k}{4} \times \left(1 - \frac{64}{64+56e^{-\frac{7}{4}kt}}\right) \\
&= \frac{7k}{4} N \left(1 - \frac{64}{1000} N\right)
\end{aligned}$$

vii. (2 marks)



$\because$ Initially when $T_1 = 120$, $N = \dfrac{1000}{120} = \dfrac{25}{3}$ and $N \geq \dfrac{25}{3}$, thus as shown in the graph above, max $\dfrac{dN}{dt}$ is when $N = \dfrac{25}{3}$.

$\therefore$ max $\dfrac{dN}{dt} = \dfrac{7}{4} \times 0.001 \times \dfrac{25}{3}\left(1 - \dfrac{64}{1000} \times \dfrac{25}{3}\right) = \dfrac{49}{7200} \approx 0.0068$ molecule/min (to 2 d.p.)

Question 14

(a) (3 marks)

- ✓ [1] for the coordinates of T, or equivalent merit
- ✓ [1] for equating $\underset{\sim}{r}$ with T, or equivalent merit
- ✓ [1] for final result

$$\underset{\sim}{r} = \begin{pmatrix} (V\cos\alpha)\,t \\ -5t^2 + (V\sin\alpha)\,t \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\because T(\mathrm{OT}\cos\beta,\ \mathrm{OT}\sin\beta)$ and equating with $\underset{\sim}{r}$,

$$\begin{aligned}
(V\cos\alpha)\,t &= \mathrm{OT}\cos\beta \quad \ldots(1) \\
-5t^2 + (V\sin\alpha)\,t &= \mathrm{OT}\sin\beta \quad \ldots(2)
\end{aligned}$$

(2) ÷ (1),

$$\begin{aligned}
\tan\beta &= \frac{-5t^2 + (V\sin\alpha)\,t}{(V\cos\alpha)\,t} \\
&= \frac{-5t + (V\sin\alpha)}{(V\cos\alpha)} \\
\therefore -5t &= V\cos\alpha\tan\beta - V\sin\alpha \\
t &= -\frac{V}{5}\left(\cos\alpha\tan\beta - \sin\alpha\right) \\
&= \frac{V}{5}\left(\sin\alpha - \cos\alpha\tan\beta\right)
\end{aligned}$$

(b) (4 marks)

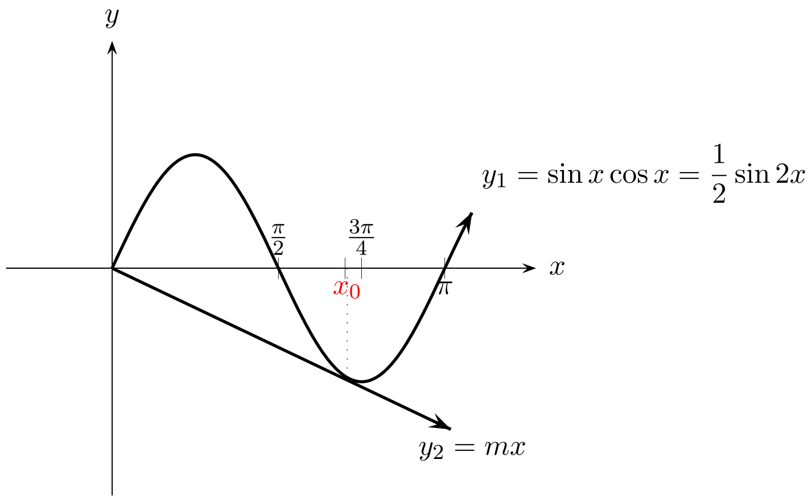
- ✓ [1] for V_{cylinder}
- ✓ [1] for integrating the expression of V_1
- ✓ [1] for evaluating V_1
- ✓ [1] for final answer

$\cos^{-1} x = \dfrac{y}{3}, x^2 = \cos^2\left(\dfrac{y}{3}\right)$

$$\begin{aligned}
V &= V_{\text{cylinder}} - V_1 \\
V_{\text{cylinder}} &= \pi(1)^2 \times 3\pi = 3\pi^2 \\
V_1 &= \pi\int_{\frac{3\pi}{2}}^{3\pi} \cos^2\left(\frac{y}{3}\right)\,dy \\
&= \frac{\pi}{2}\int_{\frac{3\pi}{2}}^{3\pi} 1 + \cos\frac{2y}{3}\,dy \\
&= \frac{\pi}{2}\left[y + \frac{3}{2}\sin\frac{2y}{3}\right]_{\frac{3\pi}{2}}^{3\pi} \\
&= \frac{\pi}{2}\left(3\pi + \frac{3}{2}\sin 2\pi - \frac{3\pi}{2} - \frac{3}{2}\sin\pi\right) \\
&= \frac{\pi}{2} \times \frac{3\pi}{2} \\
&= \frac{3\pi^2}{4} \\
\therefore V &= 3\pi^2 - \frac{3\pi^2}{4} = \frac{9\pi^2}{4}
\end{aligned}$$

(c) (4 marks)

- ✓ [1] for the two graphs shown
- ✓ [1] for showing $m = \cos(2x_0)$
- ✓ [1] for showing $\tan x_1 = x_1$
- ✓ [1] for showing $x_1 \in \left(\pi, \dfrac{3\pi}{2}\right)$



$$\frac{dy_1}{dx} = \cos 2x$$
$$\frac{dy_2}{dx} = m$$

$$\therefore \text{At } x = x_0,\ m = \cos(2x_0)$$

Let $x_1 = 2x_0$,

$$m = \cos x_1 \quad \ldots(1)$$

Also since their y-coordinates are equal,

$$\begin{aligned}
\frac{1}{2}\sin(2x_0) &= mx_0 \\
\sin 2x_0 &= m\, 2x_0 \\
\sin x_1 &= mx_1 \\
\therefore x_1 &= \frac{\sin x_1}{m} \\
&= \frac{\sin x_1}{\cos x_1} \quad \ldots \text{from } (1) \\
&= \tan x_1
\end{aligned}$$

And from the graph, $\frac{\pi}{2} < x_0 < \frac{3\pi}{4}$, $\pi < 2x_0 < \frac{3\pi}{2}$

$$\therefore x_1 \in \left(\pi, \frac{3\pi}{2}\right)$$